NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2017 HSC Course Assessment Task 4 (Trial HSC)
Monday August 7, 2017

General instructions

- Working time – 2 hours.
  (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order after this paper, tie into one bundle with the string provided and hand to examination supervisors.
- A NESA Reference Sheet is provided.

SECTION I

- Mark your answers on the answer grid provided (on page 9)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12MAT.1 – Mrs Bhamra
◯ 12MAT.2 – Mr Lam
◯ 12MAT.3 – Mr Wall
◯ 12MAT.4 – Mr Sekaran
◯ 12MAT.5 – Mrs Gan

Marker's use only.

QUESTION	1-10	11	12	13	14	Total	%
MARKS	/10	/15	/15	/15	/15	/70	

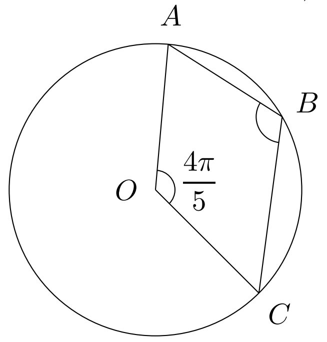
Section I

10 marks
Attempt Question 1 to 10
Allow approximately 10 minutes for this section

Mark your answers on the answer grid provided (labelled as page 9).

Questions **Marks**

1. The points A, B and C lie on a circle with centre O, as shown in the diagram. **1**



The size of $\angle AOC$ is $\dfrac{4\pi}{5}$ radians.

What is the size of $\angle ABC$ in radians?

(A) $\dfrac{3\pi}{10}$ (B) $\dfrac{\pi}{2}$ (C) $\dfrac{3\pi}{5}$ (D) $\dfrac{4\pi}{5}$

2. Which of the following is the range of the function **1**

$$y = 2\sin^{-1} x + \frac{\pi}{2}$$

(A) $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

(B) $-\dfrac{\pi}{2} \le y \le \dfrac{3\pi}{2}$

(C) $-\pi \le y \le \pi$

(D) $-\pi \le y \le \dfrac{3\pi}{2}$

3. What is the y coordinate of the point that divides the interval joining $P(-2, 2)$ and $Q(8, -3)$ internally in the ratio $3:2$? **1**

(A) 2 (B) 1 (C) 0 (D) -1

4. Which of the following is the derivative of $\sin^{-1}\dfrac{2x}{3}$? **1**

(A) $\dfrac{3}{\sqrt{9-4x^2}}$ (B) $\dfrac{2}{\sqrt{9-4x^2}}$ (C) $\dfrac{3}{\sqrt{3-2x^2}}$ (D) $\dfrac{2}{\sqrt{3-2x^2}}$

5. What is the value of $\displaystyle\lim_{x\to 0} \frac{\sin x \cos x}{2x}$? **1**

(A) 2 (B) 1 (C) $\dfrac{1}{2}$ (D) $\dfrac{1}{4}$

6. Which of the following is a correct intermediate step in evaluating **1**

$$\int_0^1 3x\left(x^2+1\right)^5\,dx$$

by using the substitution $u = x^2 + 1$?

(A) $3\int_1^2 u^5\,du$ (B) $\frac{3}{2}\int_1^2 u^5\,du$ (C) $3\int_0^1 u^5\,du$ (D) $\frac{3}{2}\int_0^1 u^5\,du$

7. A particle undergoing simple harmonic motion in a straight line has an acceleration of $\ddot{x} = 25 - 5x$, where x is the displacement after t seconds.

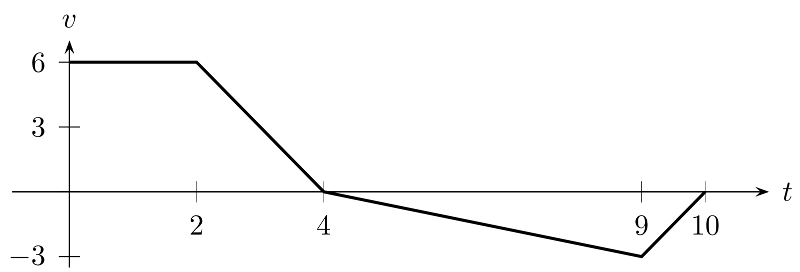
Where is the centre of motion?

(A) $x = 0$ (B) $x = 5$ (C) $x = 10$ (D) $x = 15$

8. The function $f(x) = \sin x - \frac{2x}{3}$ has a real root close to $x = 1.5$. If $x = 1.5$ is the first approximation, what is the next approximation to the root by using Newton's Method of approximation? **1**

(A) 1.495 (B) 1.496 (C) 1.503 (D) 1.504

9. The diagram below shows the velocity-time graph of an object that moves over a ten second time interval. **1**



For what percentage of time is the speed of the object decreasing?

(A) 30%

(B) 60%

(C) 70%

(D) Cannot determine from the graph.

10. What is the solution of the inequation $3x + 2 < |2x - 1|$? **1**

(A) $x < -\frac{1}{5}$

(B) $x < -3$

(C) $-3 < x < \frac{1}{5}$

(D) $x < -\frac{1}{5}$ or $x > 3$

Examination continues overleaf...

Section II

60 marks
Attempt Questions 11 to 14
Allow approximately 1 hour and 50 minutes for this section.

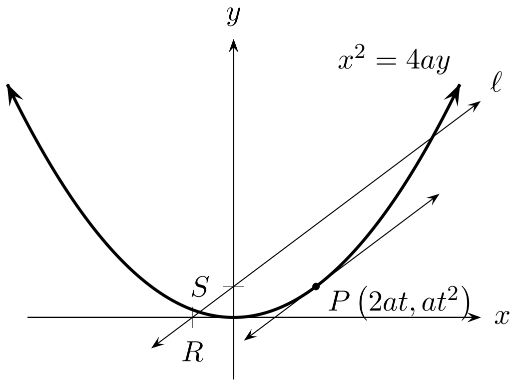
Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW booklet. **Marks**

(a) Solve for x: $\dfrac{x}{x^2-1} > 0$ **3**

(b) Find the size of the acute angle between the lines $x - y = 2$ and $2x + y = 1$. Give your answer to the nearest degree. **2**

(c) $P\left(2at, at^2\right)$ is any point on the parabola $x^2 = 4ay$. The line ℓ is parallel to the tangent at P and passes through the focus S of the parabola.



i. Find the equation of the line ℓ. **3**

ii. The line ℓ intersects the x axis at the point R. Find the coordinates of the midpoint M, of the interval RS. **2**

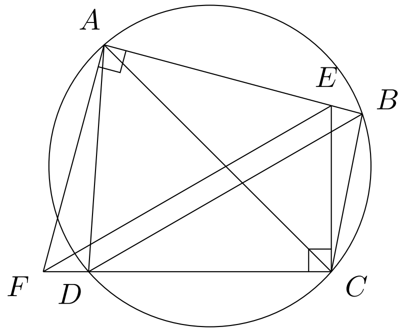
iii. Find the equation of the locus of M as the point P moves along the parabola where $t \neq 0$. **1**

(Note: Consideration of the case where $t = 0$ is not required for purposes of this question.)

Question 11 continues overleaf . . .

Question 11 continued from page 7...

(d) $ABCD$ is a cyclic quadrilateral with $\angle FAE = \angle ECD = 90°$.



Copy or trace the diagram into your writing booklet.

i. Give a reason why $AECF$ is a cyclic quadrilateral. **1**

ii. Hence prove that EF is parallel to BD. **3**

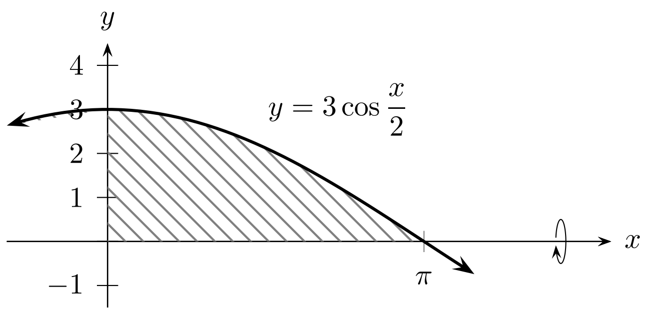
Question 12 (15 Marks) Commence a NEW booklet. **Marks**

(a) i. Show that $\cos(A - B) = \cos A \cos B\,(1 + \tan A \tan B)$. **1**

ii. Suppose that $0 < B < \dfrac{\pi}{2}$ and $B < A < \pi$. **1**

Deduce that if $\tan A \tan B = -1$, then $A - B = \dfrac{\pi}{2}$.

(b) The region bounded by the graph $y = 3\cos\dfrac{x}{2}$ and the x axis between $x = 0$ and $x = \pi$ is rotated about the x axis to form a solid. **3**



Find the exact volume of the solid.

(c) Prove by mathematical induction that $5^n + 12n - 1$ is divisible by 16 for all positive integers $n \geq 1$. **3**

(d) Find $\displaystyle\int \frac{1}{x^2 + 2x + 2}\,dx$ **2**

(e) Consider the function $f(x) = \dfrac{x}{x+4}$.

i. Explain why $f(x)$ has an inverse function $f^{-1}(x)$. **1**

ii. Find an expression for the inverse function $f^{-1}(x)$. **2**

iii. Find the point(s) of intersection of $y = f(x)$ and $y = f^{-1}(x)$. **2**

Question 13 (15 Marks) Commence a NEW booklet. **Marks**

(a) Newton's Law of Cooling states that when an object at temperature $T°$C is placed in an environment $T_0°$C, the rate of the temperature loss is given by the equation

$$\frac{dT}{dt} = -k(T - T_0)$$

where t is the time in minutes, and k is a positive constant.

An object whose initial temperature is 90°C is placed in a room in which the internal temperature is maintained at 20°C. After 10 minutes, the temperature of the object is 70°C.

i. Show that $T = T_0 + Ae^{-kt}$ satisfies the above equation. **1**

ii. Show that $k = \dfrac{1}{10}\log_e \dfrac{7}{5}$. **1**

iii. How long will it take for the object's temperature to reduce to 63°C? **2**

(b) A particle moves in a straight line such that its acceleration $\ddot{x}$ ms^{-2} is given by

$$\ddot{x} = x + \frac{3}{2}$$

Initially, the particle was 5 metres to the right of O and moving towards O with a speed of 6 ms^{-1}.

i. Initially was the particle speeding up or slowing down? Justify your answer. **1**

ii. Show that $v^2 = x^2 + 3x - 4$. **2**

iii. Where does the particle first change direction? **1**

(c) A particle moves in a straight line and its position at time t is given by

$$x = 5 + \sqrt{3}\sin 3t - \cos 3t$$

i. Express $\sqrt{3}\sin 3t - \cos 3t$ in the form $R\sin(3t - \alpha)$, where α is in radians. **2**

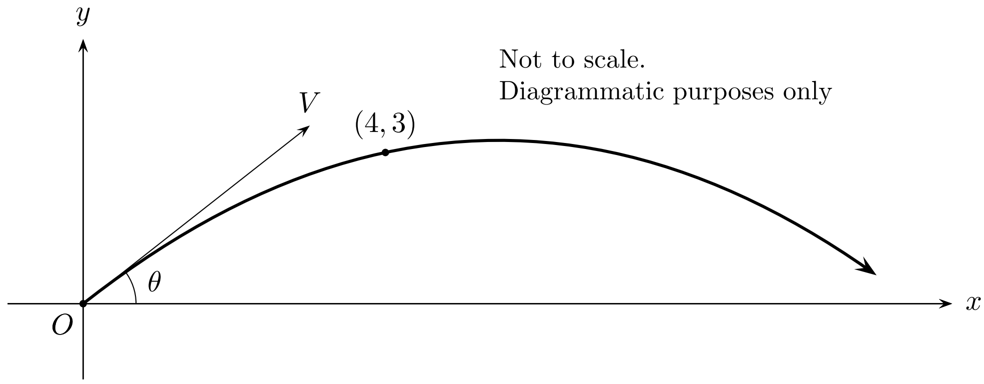
ii. Prove that the particle is undergoing simple harmonic motion. **1**

iii. Find the amplitude and centre of motion. **2**

iv. Find the first time when the particle is at its minimum displacement. **2**

Question 14 (15 Marks) Commence a NEW booklet. **Marks**

(a) A particle is projected from a point O with speed of V metres per second at an angle of θ to the horizontal. Air resistance is negligible, and the acceleration due to gravity is g ms^{-2}. The particle also passes through the point $(4, 3)$.



The displacement-time equations for the projectile are

$$x = Vt\cos\theta \qquad \text{(Do NOT prove these)}$$

$$y = -\frac{1}{2}gt^2 + Vt\sin\theta$$

i. Show that the Cartesian equation of the trajectory is given by **2**

$$y = x\tan\theta - \frac{gx^2}{2V^2}\left(1 + \tan^2\theta\right)$$

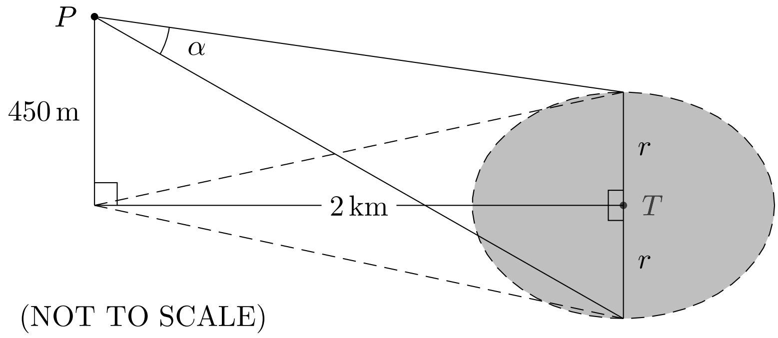
ii. Find the initial angle of projection θ to the nearest minute, if $V^2 = 8g$. **2**

iii. Find the range of the projectile. **1**

Question 14 continues overleaf . . .

Question 14 continued from page 7...

(b) An oil tanker at T is leaking oil which forms a circular oil slick. An observer is measuring the oil slick from a position P, 450 metres above sea level and 2 kilometres horizontally from the centre of the oil slick.



(NOT TO SCALE)

i. At a certain time the observer measures the angle, α , subtended by the diameter of the oil slick, to be 0.2 radians. What is the radius, r, at this time? **2**

ii. At this time, $\dfrac{d\alpha}{dt} = 0.04$ radians per hour. Find the rate at which the area of the oil slick is growing. **3**

(c) It is given that $P(x) = (x-a)^3 + (x-b)^2$. The remainder when $P(x)$ is divided by $(x-b)$ is -8.

i. Show that when $P(x)$ is divided by $(x-a)$, the remainder is 4. **1**

ii. Prove that $x = \dfrac{a+b}{2}$ is a zero of $P(x)$. **1**

iii. Prove that $P(x)$ has no stationary points. **3**

End of paper.

Sample Band E4 Responses

Section I

1. (C) **2.** (B) **3.** (D) **4.** (B) **5.** (C)
6. (B) **7.** (B) **8.** (B) **9.** (A) **10.** (A)

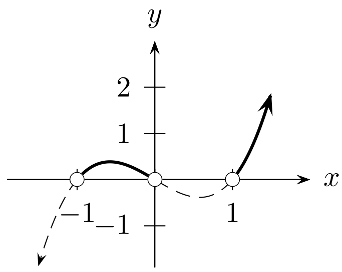
Section II

Question 11 (Lam)

(a) (3 marks)

- ✓ [1] for correctly multiplying by the square of the denominator.
- ✓ [1] diagram or otherwise in assisting with the final inequalities.
- ✓ [1] for correct inequalities.

$$\underset{\times(x^2-1)^2}{\frac{x}{x^2-1}} > \underset{\times(x^2-1)^2}{0}$$
$$x\left(x^2-1\right) > 0$$
$$x(x-1)(x+1) > 0$$



Hence $-1 < x < 0$ or $x > 1$.

(b) (2 marks)

- ✓ [1] for correct $\tan(\alpha-\beta)$ formula.
- ✓ [1] for final answer.

$$x - y = 2 \quad \Rightarrow \quad m_1 = 1$$
$$2x + y = 1 \quad \Rightarrow \quad m_2 = -2$$
$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$
$$= \frac{1-(-2)}{1+(1)(-2)}$$
$$= |-3| = 3$$
$$\therefore \theta \approx 72^\circ$$

(c) i. (3 marks)

- ✓ [1] for correctly differentiating $y = \frac{x^2}{4a}$.
- ✓ [1] for the correct gradient.
- ✓ [1] for correct equation.

$$x^2 = 4ay$$
$$y = \frac{x^2}{4a}$$
$$\frac{dy}{dx} = \left.\frac{2x}{4a}\right|_{x=2at} = \frac{4at}{4a} = t$$
$$\therefore y = tx + b$$

As the line ℓ passes through the focus $(0, a)$,

$$\therefore y = tx + a$$

ii. (2 marks)

- ✓ [1] for each correct coordinate of M.

$$y = tx + a$$

At R, $y = 0$:

$$tx + a = 0$$
$$\therefore x_R = -\frac{a}{t}$$
$$\therefore R\left(-\frac{a}{t}, 0\right)$$

The midpoint of RS:

$$M = \left(\frac{-\frac{a}{t}+0}{2}, \frac{0+a}{2}\right)$$
$$= \left(-\frac{a}{2t}, \frac{a}{2}\right)$$

iii. (1 mark)

$$y = \frac{a}{2}$$

(as y is independent of t)

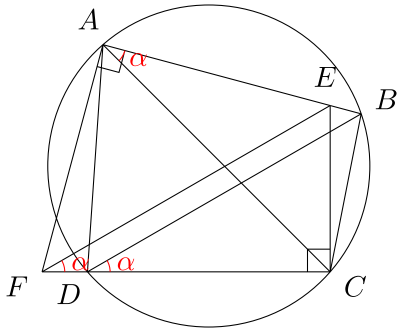
(d) i. (1 mark)

- Opposite angles $\angle FAC$ and $\angle ECF$ are supplementary (as they are both $90°$).

ii. (3 marks)

✓ [1] for finding angles in the same segment (twice)

✓ [1] for final result, based on correct argument.



- $\angle BDC = \angle BAC$ (angles in the same segment, circle $ABCD$)
- $\angle EAC = \angle EFC$ (angles in the same segment, circle $AECF$)

Hence $\angle BDC = \angle EFC$, which are corresponding angles that are equal. Hence $EF \parallel BD$.

Question 12 (Bhamra)

(a) i. (1 mark)

$$\begin{aligned}&\cos(A-B)\\&=\cos A\cos B+\sin A\sin B\\&=\cos A\cos B\left(1+\frac{\sin A\sin B}{\cos A\cos B}\right)\\&=\cos A\cos B\,(1+\tan A\tan B)\end{aligned}$$

ii. (1 mark)

$$\begin{aligned}\cos(A-B)&=\cos A\cos B\,(1+\tan A\tan B)\\&=\cos A\cos B\,(1+(-1))\\&=0\end{aligned}$$

$$\therefore A-B=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}\cdots=(2k+1)\frac{\pi}{2}$$

Since $A<\pi$ and $B>0$,

$$\therefore A-B<\pi$$
$$\therefore A-B=\frac{\pi}{2}$$

(b) (3 marks)

✓ [1] for correct volume integral.

✓ [1] for correct primitive.

✓ [1] for correct final answer.

$$\begin{aligned}V&=\pi\int_0^\pi 9\cos^2\frac{x}{2}\,dx\\&=9\pi\int_0^\pi\left(\frac{1}{2}+\frac{1}{2}\cos x\right)dx\\&=\frac{9\pi}{2}\left[x+\sin x\right]_0^\pi\\&=\frac{9\pi}{2}\left((\pi+\sin\pi)-(0+0)\right)\\&=\frac{9\pi^2}{2}\end{aligned}$$

(c) (3 marks)

✓ [1] for correctly proving the base case.

✓ [1] for using the base case in the inductive hypothesis.

✓ [1] for final proof.

Let $P(n)$ be the proposition $5^n+12n-1$ is divisible by 16, i.e.

$$5^n+12n-1=16M$$

- $P(1)$:

$$5^1+12-1=17-1=16$$

Hence $P(1)$ is true.

- Inductive step: assume $P(k)$ is true, $k\in\mathbb{Z}^+$, i.e.

$$5^k+12k-1=16P$$

is true.

- Examine $P(k+1)$:

$$\begin{aligned}&5^{k+1}+12(k+1)-1\\&=5\times5^k+12k+11\\&=5\,(16P-12k+1)+12k+11\\&=16\times5P-60k+5+12k+11\\&=16\times5P-48k+16\\&=16\,(5P-3k+1)\\&=16Q\end{aligned}$$

Hence $P(k+1)$ is also true, and $P(n)$ is true by induction.

(d) (2 marks)

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx$$
$$= \tan^{-1}(x+1) + C$$

(e) i. (1 mark)

$$f(x) = \frac{x}{x+4} = \frac{x+4-4}{x+4}$$
$$= 1 - \frac{4}{x+4}$$

i.e. a regular hyperbola which is one-to-one (monotonic increasing). Hence $f(x)$ has an inverse function.

ii. (2 marks)

✓ [1] for interchanging variables.

✓ [1] for final answer.

$$f : y = 1 - \frac{4}{x+4}$$

Interchanging variables,

$$x = 1 - \frac{4}{y+4}$$
$$x - 1 = -\frac{4}{y+4}$$
$$y + 4 = -\frac{4}{x-1}$$
$$y = -4 - \frac{4}{x-1}$$

iii. (2 marks)

✓ [1] for realising f and f^{-1} intersect along $y = x$.

✓ [1] for both points of intersection.

$f(x)$ and $f^{-1}(x)$ intersect along $y = x$:

$$x = \frac{x}{x+4}$$
$$x - \frac{x}{x+4} = 0$$
$$x\left(1 - \frac{1}{x+4}\right) = 0$$
$$x = 0 \quad \text{or} \quad \frac{1}{x+4} = 1$$
$$\therefore x = 0 \text{ or } x = -3$$

Points of intersection are $(0,0)$ and $(-3,-3)$.

Question 13 (Sekaran)

(a) i. (1 mark)

$$T = T_0 + Ae^{-kt}$$
$$\therefore T - T_0 = Ae^{-kt}$$

Differentiating,

$$\frac{dT}{dt} = -kAe^{-kt} = -k(T - T_0)$$

Hencce $T = T_0 + Ae^{-kt}$ satisfies the equation $\dfrac{dT}{dt} = -k(T - T_0)$.

ii. (1 mark)

$$t = 0, T = 90$$
$$\therefore 90 = 20 + Ae^0$$
$$A = 70$$
$$T = 20 + 70e^{-kt}$$

When $t = 10$, $T = 70$:

$$70 = 20 + 70e^{-10k}$$
$$e^{-10k} = \frac{5}{7}$$
$$-10k = \ln\frac{5}{7}$$
$$k = -\frac{1}{10}\ln\frac{5}{7} = \frac{1}{10}\ln\frac{7}{5}$$

iii. (2 marks)

✓ [1] for obtaining expression for e^{-kt}.

✓ [1] for final answer.

$$T = 20 + 70e^{-kt}$$

Finding t when $T = 63$:

$$63 = 20 + 70e^{-kt}$$
$$e^{-kt} = \frac{43}{70}$$
$$-kt = \ln\frac{43}{70}$$
$$t = \frac{1}{k}\ln\frac{70}{43}$$
$$= \frac{10\ln\frac{70}{43}}{\ln\frac{7}{5}}$$
$$\approx 14.482\cdots$$

(15 minutes)

(b) i. (1 mark)

$$\ddot{x} = x + \frac{3}{2}$$

When $t = 0$, $x = 5$:

$$\ddot{x} = 5 + \frac{3}{2} = 6.5\,\text{ms}^{-2} > 0$$
$$\dot{x} = -6\,\text{ms}^{-1} < 0$$

Hence the particle is slowing down as $\ddot{x}$ and $\dot{x}$ have opposite signs.

ii. (2 marks)

- ✓ [1] for obtaining the primitive which is equal to $\frac{1}{2}v^2$
- ✓ [1] for final answer.

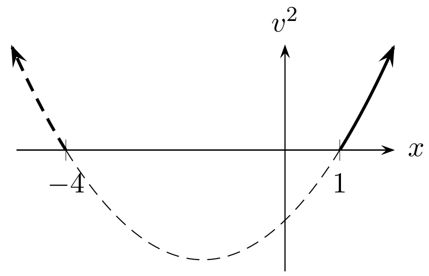
$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = x + \frac{3}{2}$$
$$\therefore \frac{1}{2}v^2 = \int \left(x + \frac{3}{2}\right)\,dx$$
$$= \frac{1}{2}x^2 + \frac{3}{2}x + C_1$$
$$v^2 = x^2 + 3x + C_2$$

When $t = 0$, $x = 5$, $v = -6$:

$$(-6)^2 = 5^2 + 3(5) + C_2$$
$$C_2 = -4$$
$$\therefore v^2 = x^2 + 3x - 4$$

iii. (1 mark)

$$v^2 = x^2 + 3x - 4 = (x-1)(x+4)$$



Particle commences at $x = 5$ and is therefore relegated to the right branch of the parabolic $v^2 - x$ graph. At $x = 1$, $v = 0$ and $\ddot{x} = 2.5 > 0$. Hence the particle first changes direction when it is 1 metre to the right of O.

(c) i. (2 marks)

$$\sqrt{3}\sin 3t - \cos 3t$$
$$= R\sin(3t - \alpha)$$
$$= R\sin 3t\cos\alpha - R\cos 3t\sin\alpha$$

Comparing coefficients,

$$\begin{cases} R\cos\alpha = \sqrt{3} & (1) \\ R\sin\alpha = 1 & (2) \end{cases}$$

(2) ÷ (1):

$$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$$
$$\alpha = \frac{\pi}{6}$$
$$R\sin\frac{\pi}{6} = 1$$
$$\therefore R = 2$$
$$\therefore \sqrt{3}\sin 3t - \cos 3t = 2\sin\left(3t - \frac{\pi}{6}\right)$$

ii. (1 mark)

$$x = 5 + \sqrt{3}\sin 3t - \cos 3t$$
$$= 5 + 2\sin\left(3t - \frac{\pi}{6}\right)$$
$$\dot{x} = 2 \times 3\cos\left(3t - \frac{\pi}{6}\right)$$
$$\ddot{x} = -2 \times 3^2\sin\left(3t - \frac{\pi}{6}\right)$$
$$= -9\left(2\sin\left(3t - \frac{\pi}{6}\right)\right)$$
$$= -9(x - 5)$$

As acceleration is proportional to but directed against the displacement from the centre of motion $x = 5$, the particle is undergoing SHM.

iii. (2 marks)

$$a = 2 \qquad x_C = 5$$

iv. (2 marks)

- ✓ [1] for recognition when the minimum displacement occurs.
- ✓ [1] for final answer.

Minimum displacement occurs when $\sin\left(3t - \frac{\pi}{6}\right) = -1$:

$$\begin{aligned}\sin\left(3t - \frac{\pi}{6}\right) &= -1\\ 3t - \frac{\pi}{6} &= \frac{3\pi}{2}, \cdots\\ 3t &= \frac{10\pi}{6} = \frac{5\pi}{3}\\ \therefore t &= \frac{5\pi}{9}\end{aligned}$$

Question 14 (Gan)

(a) i. (2 marks)

$$\begin{cases} x = Vt\cos\theta & (1)\\ y = -\frac{1}{2}gt^2 + Vt\sin\theta & (2)\end{cases}$$

From (1):

$$t = \frac{x}{V\cos\theta}$$

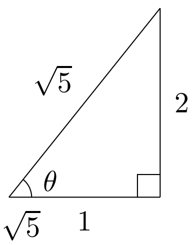
Substitute into (2):

$$\begin{aligned}y &= -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta\\ &= -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta\\ &= -\frac{gx^2}{2V^2}\left(1+\tan^2\theta\right) + x\tan\theta\end{aligned}$$

ii. (2 marks) Using $V^2 = 8g$ and substituting $x = 4$, $y = 3$:

$$\begin{aligned}3 &= -\frac{g\times 16}{2\times 8g}\left(1+\tan^2\theta\right) + 4\tan\theta\\ 3 &= -1 - \tan^2\theta + 4\tan\theta\\ \tan^2\theta - 4\tan\theta + 4 &= 0\\ (\tan\theta - 2)^2 &= 0\\ \tan\theta &= 2\\ \theta &\approx 63^\circ 26'\end{aligned}$$

iii. (1 mark) At the range R, $y = 0$. Also, draw picture depicting $\tan\theta = 2$:



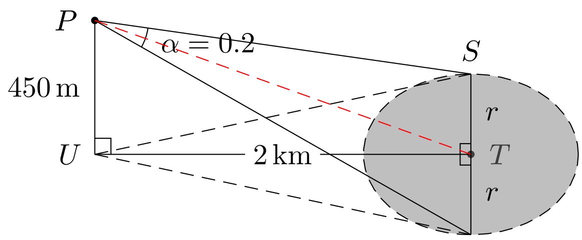
i.e. $\sec\theta = \sqrt{5}$

$$\begin{aligned}0 &= -\frac{gx^2}{16g}\sec^2\theta + x\tan\theta\\ 0 &= -\frac{x^2}{16}\times 5 + 2x\\ x\left(2 - \frac{5}{16}x\right) &= 0\end{aligned}$$

As $x \neq 0$,

$$\begin{aligned}\frac{5}{16}x &= 2\\ x &= \frac{32}{5} = 6.4\,\text{m}\end{aligned}$$

(b) i. (2 marks)



- In $\triangle PUT$,

$$\begin{aligned}PT^2 &= 450^2 + 2000^2\\ \therefore PT &= 2050\end{aligned}$$

- In $\triangle PTS$,

$$\begin{aligned}\tan 0.1 &= \frac{r}{PT} = \frac{r}{2050}\\ \therefore r &= 2050\tan 0.1 \approx 205.69\end{aligned}$$

ii. (3 marks)

- ✓ [1] for obtaining $\frac{dA}{d\alpha}$.
- ✓ [1] for substituting in values for $\frac{dA}{d\alpha}\times\frac{d\alpha}{dt}$.
- ✓ [1] for final answer.

As $A = \pi r^2$,

$$A = \pi \times 2050^2\tan^2\frac{\alpha}{2}$$

$$\begin{aligned}\frac{dA}{d\alpha} &= 2050^2\times\pi\times\sec^2\frac{\alpha}{2}\times\tan\frac{\alpha}{2}\\ &= 2050^2\times\pi\times\sec^2 0.1\times\tan 0.1\end{aligned}$$

Applying the chain rule,

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{d\alpha}\times\frac{d\alpha}{dt}\\ &= 2050^2\times\pi\times\sec^2 0.1\times\tan 0.1\times 0.04\\ &= \frac{2050^2\pi\tan 0.1}{\cos^2 0.1}\times 0.04\\ &\approx 53\,520.33 \text{ square metres per hour}\end{aligned}$$

(c) i. (1 mark)

$$\begin{aligned}P(x) &= (x-a)^3+(x-b)^2\\ P(b) &= -8\\ \therefore (b-a)^3+(b-b)^2 &= -8\\ (b-a)^3 &= -8\\ b-a &= -2 \qquad (\dagger)\end{aligned}$$

Applying the remainder theorem and evaluating $P(a)$:

$$\begin{aligned}P(a) &= (a-a)^3+(a-b)^2\\ &= (a-b)^2\\ &= 4\end{aligned}$$

Hence the remainder when divided by $(x-a)$, is 4.

ii. (1 mark)

$$\begin{aligned}P\left(\frac{a+b}{2}\right) &= \left(\frac{a+b}{2}-a\right)^3+\left(\frac{a+b}{2}-b\right)^2\\ &= \left(\frac{a+b-2a}{2}\right)^3+\left(\frac{a+b-2b}{2}\right)^2\\ &= \left(\frac{b-a}{2}\right)^3+\left(\frac{a-b}{2}\right)^2\\ &= \left(-\frac{2}{2}\right)^3+\left(\frac{2}{2}\right)^2\\ &= -1+1=0\end{aligned}$$

Hence $x=\frac{a+b}{2}$ is a zero of $P(x)$.

iii. (3 marks)

- ✓ [1] for obtaining $3(x-a)^2+2(x-b)=0$.
- ✓ [1] for correct substitution into Δ.
- ✓ [1] for correct proof.

$$P(x) = (x-a)^3+(x-b)^2$$

Differentiating,

$$P'(x) = 3(x-a)^2+2(x-b)$$

Stationary points occur when $P'(x)=0$:

$$\begin{aligned}3(x-a)^2+2(x-b) &= 0\\ 3x^2-6ax+3a^2+2x-2b &= 0\\ 3x^2+(2-6a)x+(3a^2-2b) &= 0\end{aligned}$$

Checking the discriminant of this quadratic:

$$\begin{aligned}\Delta &= (2-6a)^2-4(3)(3a^2-2b)\\ &= 4-24a+36a^2-36a^2+24b\\ &= 4-24(a-b)\\ &= 4-24(2)\\ &< 0\end{aligned}$$

$P'(x)=0$ has no real roots. Hence $P(x)$ has no stationary points.